Multicopy uncertainty observable inducing a symplectic-invariant uncertainty relation in position and momentum phase space

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We define an *uncertainty observable* acting on several replicas of a continuous-variable state, whose measurement induces phase-space uncertainty relations for a single copy of the state. By exploiting the Schwinger representation of angular momenta in terms of bosonic operators, this observable can be constructed so as to be invariant under symplectic transformations (rotation and squeezing in phase space). We first design a *two-copy* uncertainty observable, which is a discrete-spectrum operator vanishing with certainty if and only if it is applied on (two replicas of) any pure Gaussian state centered at the origin. The non-negativity of its variance translates into the Schrödinger-Robertson uncertainty relation. We then extend our construction to a *three-copy* uncertainty observable, which exhibits additional invariance under displacements (translations in phase space) so that it vanishes on every pure Gaussian state. The resulting invariance under all Gaussian unitaries makes this observable a natural tool to capture the phase-space uncertainty (or the deviation from pure Gaussianity) of continuous-variable bosonic states. In particular, it suggests that the Shannon entropy associated with the measurement of this observable provides a symplectic-invariant entropic measure of uncertainty in position-momentum phase space.

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I. INTRODUCTION

The seminal uncertainty relation due to Heisenberg [1] and more precisely formulated by Kennard [2] states that

$$\Delta x^2 \Delta p^2 \geqslant \tfrac{1}{4}, \tag{1}$$

where Δx^2 and Δp^2 are the position and momentum variances ($\hbar = 1$). The set of states that saturate this uncertainty relation are all pure Gaussian states whose covariance vanishes, i.e., those that have no x-p correlation (see [3]). Other pure Gaussian states are not minimum-uncertainty states according to the measure implied by the left-hand side of the Heisenberg relation (1) as a consequence of the fact that the latter is not invariant under rotations in phase space. This problem was solved by Schrödinger [4] and Robertson [5], who added an anticommutator term giving rise to the uncertainty relation

$$\det \gamma \geqslant \tfrac{1}{4}, \tag{2}$$

with γ being the covariance matrix. Since this determinant is invariant under symplectic transformations (rotation and squeezing) as well as displacements (translations) [6], and since it reduces to $\Delta x^2 \Delta p^2$ for states with vanishing covariance, the Schrödinger-Robertson relation (2) is saturated by all pure Gaussian states, which form the set of minimum-uncertainty states.

Variances, however, are not the only possible measure of uncertainty. In information theory, a much preferred quantity is the Shannon entropy. This measure can naturally also be applied to expressing uncertainty relations. Bialynicki-Birula and Mycielski [7] have indeed proven an entropic form of the uncertainty relation for continuous variables x and p, namely,

$$h(x) + h(p) \geqslant \ln(\pi e), \tag{3}$$

where $h(\cdot)$ stands for the Shannon differential entropy

$$h(x) = -\int p(x) \ln p(x) dx \tag{4}$$

and $p(x)$ is the probability density function of x. In some sense, entropic uncertainty relations can be considered superior to variance-based uncertainty relations. For example, it is possible to derive Eq. (1) from Eq. (3) (see [3]). The advent of quantum information theory and the special role played by entropies in this field also explains the renewed interest in entropic uncertainty relations over the past decade (see, e.g., [8–10] for recent reviews). Note that entropic uncertainty relations can be formulated for discrete variables as well, using the Shannon entropy

$$H(X) = -\sum_i p_i \ln p_i, \tag{5}$$

where p_i is the probability of measuring the outcome x_i. Here the advantage over the Heisenberg or Schrödinger-Robertson relation is the possibility to obtain a state-independent uncertainty lower bound (see, e.g., [9]).

A main drawback of the entropic uncertainty relation of Bialynicki-Birula and Mycielski is that its saturation is only reached for pure Gaussian states with zero covariance. This is because Eq. (3) is not invariant under rotations (or, more generally, symplectic transformations), exactly like Eq. (1). Recent progress has been made to define an entropic counterpart to the Schrödinger-Robertson relation [3], but no symplectic-invariant uncertainty relation that is solely expressed in terms of entropies has been found as of today. A possible, rather simple solution could be to consider the canonical pair of rotated variables $x_\theta = x\cos\theta + p\sin\theta$ and $p_\theta = -x\sin\theta + p\cos\theta$, where θ is a rotation angle. Then one could take the average or even the minimum over θ, giving, respectively,

$$\frac{1}{2\pi}\int_0^{2\pi}[h(x_\theta)+h(p_\theta)]d\theta \geqslant \ln(\pi e) \tag{6}$$

or

$$\min_\theta [h(x_\theta)+h(p_\theta)]d\theta \geqslant \ln(\pi e). \tag{7}$$

This would apparently yield two variants of a symplectic-invariant uncertainty relation based on entropies (the latter being clearly stronger than the former). However, the quantities on the left-hand side of Eqs. (6) and (7) do not appear easily tractable, so the problem remains arguably open to define a useful entropic uncertainty relation that is invariant under symplectic transformations.

In this paper we follow a path towards this goal consisting of enforcing the invariance of the measured observable instead of that of the uncertainty measure itself. We develop a framework based on the Schwinger representation of angular momenta in terms of bosonic annihilation and creation operators. This enables us to define a multicopy *uncertainty observable* with ingrained invariance under symplectic transformations in phase space (or under all Gaussian unitaries in continuous-variable state space). Then, measuring this observable allows us to express alternative uncertainty relations which logically have the appropriate invariance.

In Sec. II A we define a two-copy uncertainty observable denoted by $\hat{L}_z$, which acts on two replicas of a bosonic state and is isomorphic to the z component of an angular momentum. We present its physical representation in Sec. II B and complete it with the other two components $\hat{L}_x$ and $\hat{L}_y$ in Sec. II C. The eigensystem of $\hat{L}_z$ is then analyzed in Sec. II D, where it is shown in particular that $\hat{L}_z$ takes on (half-)integer values from $-n/2$ to $n/2$ for an n-boson system. It is invariant under symplectic transformations (rotation and squeezing) and vanishes with probability one if and only if it is applied to a Gaussian pure state that is centered at the origin in phase space. Remarkably, expressing the condition that this discrete-spectrum operator $\hat{L}_z$ has a non-negative variance translates into the Schrödinger-Robertson uncertainty relation based on the covariance matrix γ for continuous variables x and p. Then, in Sec. II E, we suggest that the Shannon entropy of $\hat{L}_z$ provides a relevant measure of uncertainty in phase space, which we compare to the Shannon differential entropy of the Wigner function in the special case of one-mode Gaussian states in Sec. II F.

Section III deals with the fact that $\hat{L}_z$ expresses an uncertainty only if applied to states centered at the origin. To overcome this limitation, we define in Secs. III A and III B a three-copy uncertainty observable denoted by $\hat{M}$, which exhibits extra invariance under displacements (Weyl operators), and hence admits all pure Gaussian states as minimum-uncertainty states. The resulting invariance under all Gaussian unitaries (rotation, squeezing, and displacement) makes this observable $\hat{M}$ a very natural measure of uncertainty in phase space (or deviation from pure Gaussianity). Its spectrum is (one-half) the spectrum of an angular momentum and, here too, the non-negativity of its variance coincides with the Schrödinger-Robertson uncertainty relation. The physical realization of the measurement of $\hat{M}$ is illustrated in Sec. III C. Then, in Sec. III D, we derive a symplectic-invariant entropic uncertainty relation based on the Shannon entropy of $\hat{M}$. It is shown that, for Gaussian states, the entropies of both multicopy observables ($\hat{L}_z$ and $\hat{M}$) are equal. The case of non-Gaussian states is also briefly discussed. We give our conclusions and summarize in Sec. IV.

II. TWO-COPY UNCERTAINTY OBSERVABLE

A. Definition of $\hat{L}_z$

Let us gain intuition on how to define an uncertainty observable. In some vague sense, we are looking for an observable that could simultaneously access both x and p quadratures.[1] To make it more precise, we consider a two-copy observable which is acting on two identical copies of state $|\psi\rangle$. Defining $|\Psi\rangle \equiv |\psi\rangle_1 \otimes |\psi\rangle_2$ as the joint state of systems 1 and 2, we may simply consider the two-copy observable $\hat{O} = \hat{x}_1 \otimes \hat{p}_2$. Its mean value gives

$$\langle\langle \hat{O} \rangle\rangle_\Psi \equiv \langle\Psi|\hat{O}|\Psi\rangle = \langle\psi|\hat{x}|\psi\rangle\langle\psi|\hat{p}|\psi\rangle. \tag{8}$$

Here and throughout this paper we use the notation $\langle\langle \hat{O} \rangle\rangle_\Psi = \langle\psi|\langle\psi|\hat{O}|\psi\rangle|\psi\rangle$ to express the mean value for two identical replicas of state $|\psi\rangle$. The second-order moment of $\hat{O}$ gives

$$\langle\langle \hat{O}^2 \rangle\rangle_\Psi = \langle\psi|\hat{x}^2|\psi\rangle\,\langle\psi|\hat{p}^2|\psi\rangle. \tag{9}$$

In the special case where the distributions of x and p are centered on zero, $\langle\langle \hat{O}^2 \rangle\rangle$ thus gives access to the product of variances $\Delta x^2\,\Delta p^2$ in state $|\psi\rangle$, which is not accessible with a single instance of the state. We may easily verify that the observable $\hat{O}$ is invariant under a squeezing of the x quadrature with parameter r, that is, under the symplectic transformation

$$\hat{x} \to \hat{x}^{(r)} = e^{-r}\hat{x}, \quad \hat{p} \to \hat{p}^{(r)} = e^{r}\hat{p}. \tag{10}$$

Indeed,

$$\hat{O}^{(r)} = \hat{x}_1^{(r)} \otimes \hat{p}_2^{(r)} = \hat{x}_1 \otimes \hat{p}_2 = \hat{O}, \tag{11}$$

so measuring $\hat{O}$ on a state $|\Psi\rangle$ is insensitive to applying a prior squeezing operation along the x (or p) quadrature on state $|\psi\rangle$. However, this property does not extend to rotated states since $\hat{O}$ is not rotation invariant.

To fix this problem, we may use instead of $\hat{O}$ the uncertainty observable defined as the two-copy operator

$$\hat{L}_z = \tfrac{1}{2}(\hat{x}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{x}_2), \tag{12}$$

where we use the index z to denote that it is the third component (or z projection) of an angular momentum $\mathbf{\hat{L}}$.

[1]From now on, we consider the x and p variables to be the canonically conjugate quadrature components of the electromagnetic field and adopt this quantum optics nomenclature. Our results, however, hold for any canonical pair of variables that is analogous to the position-momentum pair.

This definition can be motivated by taking a rotation-averaged version of the above operator $\hat{O}$. Indeed, using the symplectic transformation for a rotation of angle θ,

$$\hat{x}^{(\theta)} = \hat{x}\cos\theta + \hat{p}\sin\theta, \quad \hat{p}^{(\theta)} = -\hat{x}\sin\theta + \hat{p}\cos\theta, \quad (13)$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{x}_1^{(\theta)} \otimes \hat{p}_2^{(\theta)} d\theta = \frac{1}{2}(\hat{x}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{x}_2). \quad (14)$$

This observable is obviously invariant under rotations as well as squeezing operations; hence it is invariant under the set of all symplectic transformations.

The expectation value of $\hat{L}_z$ vanishes for all states $|\Psi\rangle$, namely,

$$\langle\langle \hat{L}_z \rangle\rangle_\Psi = \tfrac{1}{2}(\langle \hat{x}\rangle_\psi \langle \hat{p}\rangle_\psi - \langle \hat{p}\rangle_\psi \langle \hat{x}\rangle_\psi) = 0. \quad (15)$$

Its second-order moment gives

$$\begin{aligned}\langle\langle \hat{L}_z^2 \rangle\rangle_\Psi &= \tfrac{1}{2}(\langle \hat{x}^2\rangle_\psi \langle \hat{p}^2\rangle_\psi - \langle \hat{x}\hat{p}\rangle_\psi \langle \hat{p}\hat{x}\rangle_\psi)\\ &= \tfrac{1}{2}\big(\langle \hat{x}^2\rangle_\psi \langle \hat{p}^2\rangle_\psi - \tfrac{1}{4}\langle\{\hat{x},\hat{p}\}\rangle_\psi^2 + \tfrac{1}{4}\langle[\hat{x},\hat{p}]\rangle_\psi^2\big)\\ &= \tfrac{1}{2}\big(\det\gamma_c + \tfrac{1}{4}\langle[\hat{x},\hat{p}]\rangle_\psi^2\big), \quad (16)\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\langle \hat{x}\hat{p}\rangle &= \tfrac{1}{2}(\langle[\hat{x},\hat{p}]\rangle + \langle\{\hat{x},\hat{p}\}\rangle),\\ \langle \hat{p}\hat{x}\rangle &= \tfrac{1}{2}(\langle -[\hat{x},\hat{p}]\rangle + \langle\{\hat{x},\hat{p}\}\rangle).\end{aligned} \quad (17)$$

In the last line of Eq. (16), γ_c represent the covariance matrix of a state $|\psi\rangle$ centered at the origin in phase space and is defined as

$$\gamma_c = \begin{pmatrix} \langle \hat{x}^2\rangle & \tfrac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle \\ \tfrac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle & \langle \hat{p}^2\rangle \end{pmatrix} \quad (18)$$

since $\langle \hat{x}\rangle = \langle \hat{p}\rangle = 0$. Thus, the variance of our two-copy observable $(\Delta\hat{L}_z)^2 = \langle\langle \hat{L}_z^2\rangle\rangle - \langle\langle \hat{L}_z\rangle\rangle^2 = \langle\langle \hat{L}_z^2\rangle\rangle$ is linked to the determinant of the covariance matrix γ_c, namely,

$$(\Delta\hat{L}_z)^2 = \tfrac{1}{2}(\det\gamma_c + \tfrac{1}{4}\langle[\hat{x},\hat{p}]\rangle^2). \quad (19)$$

Since a variance must be non-negative, we get

$$\det\gamma_c \geqslant -\tfrac{1}{4}\langle[\hat{x},\hat{p}]\rangle^2. \quad (20)$$

If x and p are classical variables, their commutator vanishes and the symmetrization in the off-diagonal elements of γ_c has no effect; hence Eq. (20) simply implies that a classical covariance matrix is positive semidefinite. However, if $\hat{x}$ and $\hat{p}$ are canonically conjugate quantum variables, they do not commute ($[\hat{x},\hat{p}] = i$) and Eq. (20) is nothing but the Schrödinger-Robertson uncertainty relation $\det\gamma \geqslant \frac{1}{4}$, where γ denotes the usual covariance matrix of a state.[2] From this perspective, the Schrödinger-Robertson uncertainty relation simply expresses the inequality $\langle\langle \hat{L}_z^2\rangle\rangle \geqslant 0$, where we first need to center the state before measuring $\hat{L}_z$. In some sense, this inequality may be deemed trivial as it expresses the fact that the variance of an operator is non-negative. However, its equivalence with the Schrödinger-Robertson uncertainty relation suggests an alternate formulation of the uncertainty relation in terms of the entropy of $\hat{L}_z$, as analyzed in Sec. II E.

[2]Indeed, the covariance matrix γ as defined in Eq. (61) is invariant under displacements, which means that $\det\gamma = \det\gamma_c$ for a state centered on the origin.

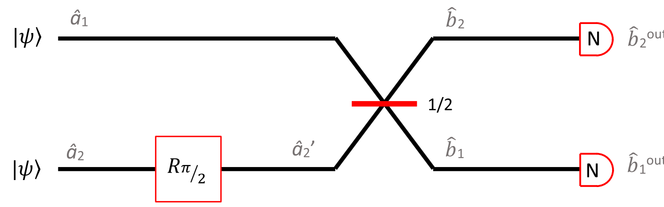

FIG. 1. Physical realization of a measurement of the two-copy uncertainty observable $\hat{L}_z$. Starting from two identical copies of state $|\psi\rangle$, we apply a $\pi/2$ phase rotation on the second mode and then process the two modes via a 50:50 beam splitter. By measuring the photon-number difference of the output state, we access $\hat{L}_z$. The outcome is zero if and only if $|\psi\rangle$ is a minimum-uncertainty state (Gaussian pure state centered on the origin in phase space).

B. Physical realization of $\hat{L}_z$

Let us give a physical interpretation to the two-copy uncertainty observable $\hat{L}_z$. By using the mode operators $\hat{a}_j = (\hat{x}_j + i\hat{p}_j)/\sqrt{2}$ for $j = 1, 2$, we may rewrite it as

$$\hat{L}_z = \frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2). \quad (21)$$

From this definition it is easy to confirm that the action of $\hat{L}_z$ gives 0 on any pure Gaussian state centered on the origin, i.e., any squeezed vacuum state. Let $|s\rangle = S(s)|0\rangle$ denote a squeezed vacuum state, where $|0\rangle$ is the vacuum state and $S(s) = e^{(s^*\hat{a}^2 - s\hat{a}^{\dagger 2})/2}$ is the squeezing operator with the parameter $s = re^{i\phi}$. Using $\hat{a}|0\rangle = 0 \;\Leftrightarrow S(s)\hat{a}S^\dagger(s)|s\rangle = 0$ $\Leftrightarrow [\cosh(r)\hat{a} + e^{i\phi}\sinh(r)\hat{a}^\dagger]|s\rangle = 0$, we see that $|s\rangle$ satisfies $\hat{a}|s\rangle = -e^{i\phi}\tanh(r)\hat{a}^\dagger|s\rangle$. Therefore,

$$\begin{aligned}\hat{L}_z|s\rangle|s\rangle &= \frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2)|s\rangle|s\rangle\\ &= \frac{i}{2}\{[-e^{i\phi}\tanh(r)\hat{a}_1^\dagger]\hat{a}_2^\dagger - \hat{a}_1^\dagger[-e^{i\phi}\tanh(r)\hat{a}_2^\dagger]\}|s\rangle|s\rangle\\ &= 0. \quad (22)\end{aligned}$$

More interestingly, this formulation of $\hat{L}_z$ provides us with a nice physical interpretation of the uncertainty observable in terms of a beam-splitter transformation. As shown in Fig. 1, if we make a $\pi/2$ phase rotation on the second mode, $\hat{a}_2 \to \hat{a}_2' = -i\hat{a}_2$, followed by a 50:50 beam-splitter transformation of the two modes according to

$$\hat{a}_1 \to \hat{b}_1 = (\hat{a}_1 + \hat{a}_2')/\sqrt{2}, \quad \hat{a}_2' \to \hat{b}_2 = (\hat{a}_1 - \hat{a}_2')/\sqrt{2}, \quad (23)$$

we may reexpress the uncertainty observable as

$$\hat{L}_z = \tfrac{1}{2}(\hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2), \quad (24)$$

where $\hat{b}_1$ and $\hat{b}_2$ denote the output mode operators. Thus, $\hat{L}_z$ corresponds (up to a factor $\frac{1}{2}$) to the difference between the photon numbers at the two output modes of the beam splitter, that is, $\hat{L}_z = (\hat{n}_1^{out} - \hat{n}_2^{out})/2$.

Recall that a two-mode squeezed vacuum state can be realized with two single-mode squeezed vacuum states with orthogonal squeezing orientations followed by a 50:50 beam splitter. Thus, if we start with two identical replicas of an arbitrary squeezed vacuum state $|s\rangle|s\rangle$ and rotate one of them by an angle $\pi/2$ before processing both of them through a 50:50 beam splitter, we get precisely a two-mode squeezed vacuum state. Such a state exhibits perfect photon-number correlations since it is written as $\sum_n c_n|n\rangle|n\rangle$, so measuring the photon-number difference gives zero with certainty. This is consistent with the fact that our observable $\hat{L}_z$ gives a value 0 and exhibits no uncertainty (zero variance) when applied to any pure Gaussian state centered on the origin. We have thus found a simple experimentally relevant method for measuring the uncertainty of a state (or its deviation with respect to a pure Gaussian state[3]).

C. Algebra of angular momenta $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$

By exploiting the analogy with the algebra of angular momenta, it is possible to define the two-copy operators $\hat{L}_x$ and $\hat{L}_y$, which in turn allows us to define the ladder operators $\hat{L}_+$ and $\hat{L}_-$. The definition of $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$ follows from the Schwinger representation, which yields a connection between an angular momentum and two uncoupled harmonic oscillators (or bosonic modes) [11]. In quantum optics, it is also linked to the definition of the Stokes operators in the description of the polarization of light [12–14]. The easiest way to proceed is to note that $\hat{L}_z$ as defined in Eq. (21) can be reexpressed as

$$\hat{L}_z = \tfrac{1}{2}\hat{A}^\dagger\sigma_y\hat{A}, \tag{25}$$

where $\hat{A} = \binom{\hat{a}_1}{\hat{a}_2}$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$ is the second Pauli matrix. Similarly, we can define

$$\hat{L}_y = \tfrac{1}{2}\hat{A}^\dagger\sigma_x\hat{A}, \quad \hat{L}_x = \tfrac{1}{2}\hat{A}^\dagger\sigma_z\hat{A}, \tag{26}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ are the other two Pauli matrices. In terms of mode operators or quadrature operators, this gives

$$\begin{aligned}\hat{L}_y &= \tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger)\\ &= \tfrac{1}{2}(\hat{x}_1\hat{x}_2 + \hat{p}_1\hat{p}_2),\\ \hat{L}_x &= \tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)\\ &= \tfrac{1}{4}\big[(\hat{x}_1^2 + \hat{p}_1^2) - (\hat{x}_2^2 + \hat{p}_2^2)\big]\\ &= \tfrac{1}{2}(\hat{n}_1 - \hat{n}_2).\end{aligned} \tag{27}$$

Since the Pauli matrices respect the commutation relation $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, where ϵ_{ijk} is the Levi-Cività symbol, it can be verified that our three two-copy operators respect the commutation relations for angular momenta (see Appendix A)

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k. \tag{28}$$

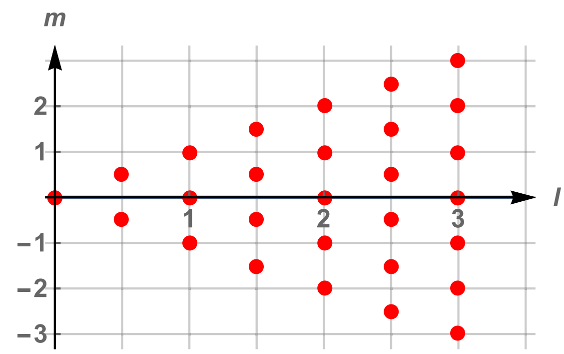

FIG. 2. Possible eigenvalues m of a state $\|l, m\rangle\rangle$ with a total photon number equal to $2l$.

We can then define the ladder operators

$$\begin{aligned}\hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \tfrac{1}{2}(\hat{a}_1^\dagger + i\hat{a}_2^\dagger)(\hat{a}_1 + i\hat{a}_2),\\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y = \tfrac{1}{2}(\hat{a}_1^\dagger - i\hat{a}_2^\dagger)(\hat{a}_1 - i\hat{a}_2),\end{aligned} \tag{29}$$

as well as the squared angular momentum operator

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}_0(\hat{L}_0 + 1), \tag{30}$$

where

$$\hat{L}_0 = \tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) = \tfrac{1}{2}(\hat{n}_1 + \hat{n}_2) \tag{31}$$

is the Casimir operator.

The definitions of $\hat{L}_z$ given in Eqs. (21) and (24) also suggest that all three angular momentum components $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$ can be expressed in alternative ways as a function of the input mode operators $(\hat{a}_1, \hat{a}_2)$, output mode operators $(\hat{b}_1, \hat{b}_2)$, or even the output mode operators of another circuit $(\hat{c}_1, \hat{c}_2)$. This is summarized in Appendix B, together with the corresponding physical realizations of $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$.

D. Eigensystem of $\hat{L}_z$

In order to calculate the Shannon entropy of the uncertainty observable $\hat{L}_z$, we need first to determine the eigensystem of this operator. Defining $l = (n_1 + n_2)/2$, we see from Eqs. (30) and (31) that the eigenvalues of $\hat{L}^2$ are given by $l(l+1)$, just as the eigenvalues of the squared modulus of an angular momentum. Thus, we may label the eigenvectors of $\hat{L}_z$ and $\hat{L}^2$ by $\|l, m\rangle\rangle$, where l represents one-half of the total photon number and m is the eigenvalue of $\hat{L}_z$ (with $|m| \leqslant l$), so that

$$\begin{aligned}\hat{L}_z\|l, m\rangle\rangle &= m\|l, m\rangle\rangle,\\ \hat{L}^2\|l, m\rangle\rangle &= l(l+1)\|l, m\rangle\rangle,\\ \hat{L}_\pm\|l, m\rangle\rangle &= \sqrt{l(l+1) - m(m\pm 1)}\|l, m\pm 1\rangle\rangle.\end{aligned} \tag{32}$$

Given the commutation relations (28), the possible eigenvalues of $\hat{L}_z$ for every value of l are $m \in \{-l, l\}$ with integer jumps[4] as sketched in Fig. 2. The eigenvectors of $\hat{L}_z$ and $\hat{L}^2$

[3]This method is limited to states centered at the origin in phase space, but we will show in Sec. III how it can be generalized to all states.

[4]Indeed, $[\hat{L}_z, \hat{L}_+] = \hat{L}_+$ and thus $\hat{L}_z\hat{L}_+\|l, m\rangle\rangle = (\hat{L}_+\hat{L}_z + \hat{L}_+)\|l, m\rangle\rangle = (m+1)\hat{L}_+\|l, m\rangle\rangle$, where we assumed that $\hat{L}_z\|l, m\rangle\rangle = m\|l, m\rangle\rangle$.

can be expressed, in general, as linear combinations of the two-mode Fock states $|j, k\rangle$,

$$\| l, m\rangle\rangle = \sum_j \sum_k c_{jk} |j, k\rangle. \quad (33)$$

When fixing the value of l, the only nonzero c_{jk}'s are of course those such that $j + k = 2l$. Let us start with examples for some specific values of l. If we set $l = 0$, the only eigenvector is

$$\|0, 0\rangle\rangle = |0, 0\rangle. \quad (34)$$

If we set $l = \frac{1}{2}$, we have two eigenvectors with eigenvalues $m = \pm\frac{1}{2}$, namely,

$$\begin{aligned} \|\tfrac{1}{2}, -\tfrac{1}{2}\rangle\rangle &= \frac{1}{\sqrt{2}}(|0, 1\rangle + i|1, 0\rangle), \\ \|\tfrac{1}{2}, \tfrac{1}{2}\rangle\rangle &= \frac{1}{\sqrt{2}}(|0, 1\rangle - i|1, 0\rangle). \end{aligned} \quad (35)$$

If we set $l = 1$, we have three eigenvectors with eigenvalues $m = \{-1, 0, 1\}$, namely,

$$\begin{aligned} \|1, -1\rangle\rangle &= \frac{1}{2}(|2, 0\rangle - i\sqrt{2}|1, 1\rangle - |0, 2\rangle), \\ \|1, 0\rangle\rangle &= \frac{1}{\sqrt{2}}(|2, 0\rangle + |0, 2\rangle), \\ \|1, 1\rangle\rangle &= \frac{1}{2}(|2, 0\rangle + i\sqrt{2}|1, 1\rangle - |0, 2\rangle). \end{aligned} \quad (36)$$

For higher values of l, it becomes cumbersome to write the general form of the eigenstates, but we can in principle construct them by applying the ladder operator $\hat{L}_+$. We start from the eigenstates corresponding to the lowest diagonal in Fig. 2, that is, states $\|l, -l\rangle\rangle$ whose (unnormalized) form is defined as

$$\begin{aligned} \|l, -l\rangle\rangle = &\sum_{k=0}^{\lfloor l-1/2 \rfloor} i^k \sqrt{\binom{2l}{k}} [|k, 2l-k\rangle + (-1)^k i^{2l} |2l-k, k\rangle] \\ &+ \frac{1 + (-1)^{2l}}{2} i^l \sqrt{\binom{2l}{l}} |l, l\rangle. \end{aligned} \quad (37)$$

We simply need to apply repeatedly the operator $\hat{L}_+$ as defined in Eq. (29) in order to find all other eigenstates, since

$$\|l, m+1\rangle\rangle = \frac{1}{\sqrt{l(l+1) - m(m+1)}} \hat{L}_+ \|l, m\rangle\rangle. \quad (38)$$

We thus have access to all eigenstates $\|l, m\rangle\rangle$.

Returning to the interpretation of $\hat{L}_z$ as an uncertainty observable, let us discuss the special case of an even total photon number, i.e., when l is an integer. In this case, there is always an eigenstate that admits the eigenvalue $m = 0$. Its general (unnormalized) form is

$$\begin{aligned} \|l, 0\rangle\rangle = &\beta \frac{1 + (-1)^l}{2} |l, l\rangle \\ &+ \sum_{i=0}^{\lfloor l/2 - 1/2 \rfloor} \alpha_i (|2i, 2l-2i\rangle + |2l-2i, 2i\rangle), \end{aligned} \quad (39)$$

with

$$\begin{aligned} \alpha_i &= \sqrt{\frac{(2l)!!(2l-2i-1)!!(2i-1)!!}{(2l-2i)!!(2l-1)!!(2i)!!}}, \\ \beta &= \sqrt{\frac{(2l)!!(l-1)!!(l-1)!!}{(l)!!(2l-1)!!(l)!!}}, \end{aligned} \quad (40)$$

where $(\cdot)!!$ denotes the double factorial and the index i is an integer. This means that the states $\|l, 0\rangle\rangle$ are thus written as linear combinations involving only even Fock states of the form $|2j, 2k\rangle$. This is connected to the fact that a squeezed vacuum state only involves even Fock states in its expansion. Taking two copies of a squeezed vacuum state $|s\rangle$, namely,

$$|s\rangle \otimes |s\rangle = \frac{1}{\cosh r} \sum_{j,k=0}^{\infty} \frac{\sqrt{(2j)!!(2k)!!}}{2^{j+k} j! k!} (\tanh r)^{k+j} |2j, 2k\rangle, \quad (41)$$

we get again a linear combination of even Fock states $|2j, 2k\rangle$. This implies that $|s\rangle \otimes |s\rangle$ can be expressed as a linear combination of eigenstates $\|l, 0\rangle\rangle$ (with l integer). Therefore, applying $\hat{L}_z$ on $|s\rangle \otimes |s\rangle$ gives zero, which confirms that all squeezed vacuum states $|s\rangle$ are minimum-uncertainty states for the uncertainty observable $\hat{L}_z$ in accordance with Eq. (22).

Finally, let us mention an interesting symmetry property of the eigenstates $\|l, m\rangle\rangle$ with respect to the exchange operator $\hat{P}$, which exchanges the indices of the systems 1 and 2. This operator can be seen as a reflection along the $x_1 = x_2$ line and $p_1 = p_2$ line in phase space and it acts on $\hat{L}_z$, $\hat{L}_y$, and $\hat{L}_x$ as

$$\hat{P}\hat{L}_z\hat{P} = -\hat{L}_z, \quad \hat{P}\hat{L}_y\hat{P} = \hat{L}_y, \quad \hat{P}\hat{L}_x\hat{P} = -\hat{L}_x, \quad (42)$$

where we used $\hat{P}^\dagger = \hat{P}$. Note also that $\hat{P}\hat{L}_\pm\hat{P} = -\hat{L}_\mp$. Hence, we can evaluate the action of $\hat{P}$ on the eigenstates of $\hat{L}_z$. Since $\hat{L}_z\|l, m\rangle\rangle = m\|l, m\rangle\rangle$ we have

$$\begin{aligned} &-\hat{P}\hat{L}_z\hat{P}\|l, m\rangle\rangle = m\|l, m\rangle\rangle \\ &\Leftrightarrow \hat{L}_z\hat{P}\|l, m\rangle\rangle = -m\hat{P}\|l, m\rangle\rangle, \end{aligned} \quad (43)$$

where we used $\hat{P}^{-1} = \hat{P}$. Thus, $\hat{P}\|l, m\rangle\rangle$ is proportional to the eigenstate of $\hat{L}_z$ with eigenvalue $-m$, namely,

$$\hat{P}\|l, m\rangle\rangle \propto \|l, -m\rangle\rangle. \quad (44)$$

Starting from eigenstate $\|l, m\rangle\rangle$, we obtain the eigenstate $\|l, -m\rangle\rangle$ simply by interchanging systems 1 and 2. From Eq. (44) we also understand that the states $\|l, 0\rangle\rangle$ must be symmetric under the exchange of both systems, as can be checked from Eq. (39).

E. Entropic uncertainty relation based on $\hat{L}_z$

We saw in Sec. II A that the non-negativity of the variance of our uncertainty observable $\hat{L}_z$ coincides with the Schrödinger-Robertson uncertainty relation (for states centered at the origin). We will now turn to the Shannon entropy of $\hat{L}_z$ and show that it provides a relevant symplectic-invariant measure of uncertainty. Since we know the eigensystem of $\hat{L}_z$ (see Sec. II D), we can in principle compute its Shannon entropy [as defined in Eq. (5)], that is,

$$H(\hat{L}_z)_\rho = -\sum_m p_m \ln p_m, \quad (45)$$

where p_m is the probability of measuring eigenvalue m (which goes from $-\infty$ to ∞ in steps of $\frac{1}{2}$) when having two replicas of state ρ, namely,

$$p_m = \sum_{l=|m|}^{\infty} \langle\langle l, m\| \rho \otimes \rho \|l, m\rangle\rangle. \quad (46)$$

The sum over l starts at $l = |m|$ since $-l \leqslant m \leqslant l$ and includes only integer (half-integer) values if m is an integer (a half-integer).

Just like the variance, the Shannon entropy is a nonnegative quantity, so it is natural to write

$$H(\hat{L}_z)_\rho \geqslant 0, \quad (47)$$

which is the entropic counterpart of the Schrödinger-Robertson uncertainty relation $\langle\langle \hat{L}_z^2 \rangle\rangle \geqslant 0$. It is saturated by all pure Gaussian states (centered on the origin) and is invariant under symplectic transformations, i.e., under any Gaussian unitary except displacements.

Indeed, suppose we apply $U \otimes U$ on an eigenstate $\|l, m\rangle\rangle$, where U is such a Gaussian unitary. Since $\hat{L}_z$ is invariant under U, i.e., $U^\dagger \otimes U^\dagger \hat{L}_z U \otimes U = \hat{L}_z$, we have

$$\hat{L}_z U \otimes U \|l, m\rangle\rangle = m U \otimes U \|l, m\rangle\rangle, \quad (48)$$

so $U \otimes U \|l, m\rangle\rangle$ is an eigenvector of $\hat{L}_z$ with the same eigenvalue m. Thus, the eigenspace spanned by all states with eigenvalue m is invariant under $U \otimes U$. Hence, the projector associated with the measurement of outcome m,

$$\mathbb{P}_m = \sum_{l=|m|}^{\infty} \|l, m\rangle\rangle\langle\langle l, m\|, \quad (49)$$

is invariant under $U \otimes U$, and so is the probability of measuring m, namely, $p_m = \mathrm{Tr}(\rho \otimes \rho\, \mathbb{P}_m)$. Therefore, the Shannon entropy $H(\hat{L}_z)_\rho$ is invariant under symplectic transformations, as advertised.

F. Special case of Gaussian states

Although it should be easy to measure $\hat{L}_z$ experimentally (with the circuit in Fig. 1) and then compute its Shannon entropy, it does not seem straightforward to calculate $H(\hat{L}_z)$ analytically for a given state $|\psi\rangle$ because one needs first to express $\|\Psi\rangle\rangle$ as a linear combination of the eigenstates $\|l, m\rangle\rangle$. The calculation of $H(\hat{L}_z)$ for some simple examples of non-Gaussian states is illustrated in Appendix C. However, this calculation does not require much effort in the special case of Gaussian states (centered on the origin). Beforehand, recall that, according to Williamson theorem, every Gaussian state can be brought to a thermal state by applying some Gaussian unitary [6]. Since $H(\hat{L}_z)$ is invariant under Gaussian unitaries,[5] it is enough to compute its value for a thermal state (it is then the same for any Gaussian state with the same symplectic spectrum). Luckily, it is straightforward to evaluate $H(\hat{L}_z)$ for a thermal state

$$\rho_{th} = \sum_{n=0}^{\infty} \frac{\langle n\rangle^n}{(\langle n\rangle + 1)^{n+1}} |n\rangle\langle n| \quad (50)$$

because when inserting $\rho_{th} \otimes \rho_{th}$ in the circuit of Fig. 1, measuring $\hat{L}_z$ simply corresponds to measuring the difference between the photon numbers at the two outputs, $\hat{d} = (\hat{n}_1^{\text{out}} - \hat{n}_2^{\text{out}})/2$. Since a thermal state is invariant under rotation in phase space, the second mode remains in state ρ_{th} after the $\pi/2$ rotation shown in Fig. 1. Moreover, when two copies of a thermal state are inserted in a beam splitter, the output is again the product of the same two thermal states. The random variable d is just the difference of two independent (geometrically distributed) random variables. The probability of measuring n_i photons on the ith output mode is

$$P(\hat{n}_i = n_i) = \frac{\langle n\rangle^{n_i}}{(\langle n\rangle + 1)^{n_i+1}}, \quad i = 1, 2, \quad (51)$$

so the probability of obtaining a certain value for the (half) difference d is

$$P(\hat{d} = d) = \begin{cases} \sum_{n_2=0}^{\infty} P(\hat{n}_1 = n_2 + 2d)P(\hat{n}_2 = n_2), & d > 0 \\ \sum_{n_1=0}^{\infty} P(\hat{n}_1 = n_1)P(\hat{n}_2 = n_1 - 2d), & d < 0 \\ \sum_{n_1=0}^{\infty} P(\hat{n}_1 = n_1)P(\hat{n}_2 = n_1), & d = 0. \end{cases} \quad (52)$$

This yields

$$P(\hat{d} = d) = \frac{1}{2\langle n\rangle + 1}\left(\frac{\langle n\rangle}{\langle n\rangle + 1}\right)^{2d}, \ \forall d. \quad (53)$$

We can now compute the Shannon entropy of $\hat{L}_z$ as

$$H(\hat{L}_z)_{\rho_{th}} = -\sum_d P(\hat{d} = d) \ln P(\hat{d} = d) = \ln(2\langle n\rangle + 1) + E(\langle n\rangle), \quad (54)$$

where

$$E(\langle n\rangle) = -\frac{2\langle n\rangle(\langle n\rangle + 1)}{2\langle n\rangle + 1} \ln \frac{\langle n\rangle}{\langle n\rangle + 1} \quad (55)$$

is a function ranging between 0 and 1, as plotted in Fig. 3. Note that d can be an integer or a half-integer in Eq. (53)

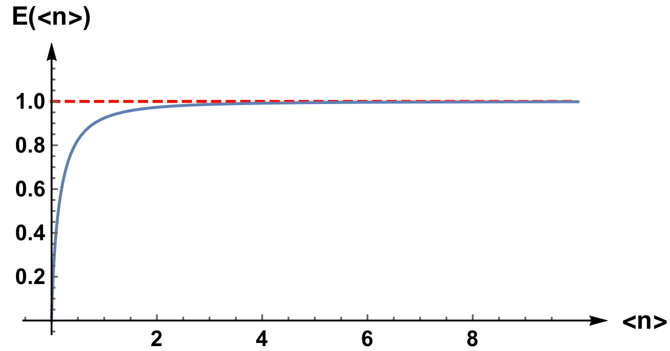

FIG. 3. Graph of $E(\langle n\rangle)$.

[5] We only consider Gaussian states centered at the origin, which can be brought to a thermal state by applying a symplectic transformation (no displacement is needed), so the invariance of $H(\hat{L}_z)$ holds.

and this must be taken into account when summing over d in Eq. (54).

Interestingly, if we compute the Shannon differential entropy[6] $h(x, p)$ of a thermal state with the Wigner function

$$W_{\rho_{th}} = \frac{1}{2\pi\sqrt{\det\gamma}} \exp\left[-\frac{1}{2}(x \quad p)^T \gamma^{-1}\begin{pmatrix} x \\ p \end{pmatrix}\right] = \frac{1}{\pi(2\langle n\rangle + 1)} \exp\left(-\frac{1}{2\langle n\rangle + 1}(x^2 + p^2)\right), \quad (56)$$

we find

$$h(x, p)_{\rho_{th}} = -\int W_{\rho_{th}}(x, p) \ln W_{\rho_{th}}(x, p) dx\, dp = \ln(\pi e) + \ln(2\langle n\rangle + 1) \quad (57)$$

which implies that

$$H(\hat{L}_z)_{\rho_{th}} = h(x, p)_{\rho_{th}} - \ln(\pi e) + E(\langle n\rangle). \quad (58)$$

This expression is interesting as it combines the Shannon entropy of our discrete uncertainty observable $\hat{L}_z$ with the Shannon differential entropy of two continuous variables x and p. The first term on the right-hand side of Eq. (57) is the Shannon differential entropy of the Wigner function for the vacuum state $h(x, p)_{\rho_{vac}} = \ln(\pi e)$, so Eq. (58) implies that $H(\hat{L}_z)_{\rho_{th}}$ is close to $h(x, p)_{\rho_{th}} - h(x, p)_{\rho_{vac}}$ within a range of $0 \leqslant E(\langle n\rangle) \leqslant 1$. This is a way of understanding Eq. (47) as an entropic uncertainty relation, measuring the distance from a pure Gaussian state (here the vacuum state).

To be complete, let us also express the above entropies in terms of the symplectic value ν, so this applies to any Gaussian state ρ_G. Using the fact that $\langle n\rangle = \nu - \frac{1}{2}$ for thermal states, we get

$$H(\hat{L}_z)_{\rho_G} = \ln(2\nu) - \frac{4\nu^2 - 1}{4\nu} \ln\frac{2\nu - 1}{2\nu + 1}, \quad (59)$$

$$h(x, p)_{\rho_G} = \ln(\pi e) + \ln(2\nu). \quad (60)$$

Note that $H(\hat{L}_z)_{\rho_G}$ is monotonically increasing in ν. The only thermal state that has $H(\hat{L}_z) = 0$ is the vacuum state (considering states centered on the origin). Equivalently, all pure Gaussian states ($\nu = \frac{1}{2}$) saturate our entropic uncertainty relation (47), and the quantity $H(\hat{L}_z)$ can be seen as a measure of pure non-Gaussianity. Finally, if we only consider Gaussian states, $H(\hat{L}_z)$ as defined in Eq. (59) may also be understood as a measure of mixedness since the purity of a Gaussian state is given by $\mu = \mathrm{Tr}\rho_G^2 = \frac{1}{2}\nu$.

III. THREE-COPY UNCERTAINTY OBSERVABLE

A. Definition of $\hat{M}$

The two-copy operator $\hat{L}_z$ expresses the uncertainty solely for states centered at the origin. To overcome this limitation, we define a three-copy uncertainty observable, denoted by $\hat{M}$ in the following. The intuition comes from Ref. [15], where it is shown that any nth-degree polynomial function of the elements of a single-copy density matrix ρ can be computed as the expectation value of some well-chosen n-copy observable acting on $\rho^{\otimes n}$.

We define the covariance matrix γ for any state, not necessarily centered on 0, as

$$\gamma = \begin{pmatrix} \langle x^2\rangle - \langle x\rangle^2 & \frac{1}{2}\langle\{x, p\}\rangle - \langle x\rangle\langle p\rangle \\ \frac{1}{2}\langle\{x, p\}\rangle - \langle x\rangle\langle p\rangle & \langle p^2\rangle - \langle p\rangle^2 \end{pmatrix}. \quad (61)$$

This definition is valid for both classical and quantum variables. If we compute its determinant, we then have

$$\det\gamma = \langle x^2\rangle\langle p^2\rangle - \langle x^2\rangle\langle p\rangle\langle p\rangle - \langle p^2\rangle\langle x\rangle\langle x\rangle - \tfrac{1}{4}\langle\{x, p\}\rangle^2 + \langle\{x, p\}\rangle\langle x\rangle\langle p\rangle. \quad (62)$$

From Ref. [15] we thus know that this expression must in principle be writable as the expectation value of some four-copy observable. Here we will show that a three-copy observable $\hat{M}$ is actually sufficient if we consider its variance (rather than its expectation value) and follow a similar procedure as for the two-copy observable $\hat{L}_z$. As we have seen, the latter is the z component of an angular momentum in the Schwinger representation, but the other two components $\hat{L}_x$ and $\hat{L}_y$ are not linked to uncertainty. In contrast, here we treat the three components $\hat{M}_i$ of an angular momentum on an equal footing and define[7]

$$\hat{M}_x = \tfrac{1}{2}(\hat{x}_2\hat{p}_3 - \hat{p}_2\hat{x}_3),$$
$$\hat{M}_y = \tfrac{1}{2}(\hat{x}_3\hat{p}_1 - \hat{p}_3\hat{x}_1), \quad (63)$$
$$\hat{M}_z = \tfrac{1}{2}(\hat{x}_1\hat{p}_2 - \hat{p}_1\hat{x}_2).$$

The three-copy uncertainty observable reads

$$\hat{M} = \frac{1}{\sqrt{3}}(\hat{M}_x + \hat{M}_y + \hat{M}_z) \quad (64)$$

and can be viewed as the projection of the angular momentum $\hat{\mathbf{M}}$ onto a line halfway between the x, y, and z axes. Since the two-copy observable $\hat{L}_z$ is invariant under symplectic transformations (rotations and squeezing), so are all the $\hat{M}_i$ observables since they have the same form as $\hat{L}_z$ acting on two of the three copies. Hence, the three-copy observable $\hat{M}$ is also invariant under symplectic transformations. Furthermore, $\hat{M}$ is this time also invariant under displacements. Indeed, since we consider three copies of the same state, the displacement is the same in each of the three modes. In other words, the displacement in position x (or momentum p) is always applied in the direction $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, which is exactly the direction of the angular momentum component $\hat{M}$. Since the projection of an angular momentum along a direction is invariant under a position shift (or a momentum kick) in that direction, $\hat{M}$ is invariant under displacements, so we have relaxed the need to restrict to states centered at the origin.

Interestingly, the variance of $\hat{M}$ can be related to the determinant of the covariance matrix γ exactly as we had done for $\hat{L}_z$ in Sec. II A. First, we remark that $\langle\langle\langle\hat{M}\rangle\rangle\rangle_\Psi = 0$, where

[6] Since a thermal state has a positive Wigner function, its Shannon differential entropy is simply the classical entropy of the joint probability distribution of (x, p) given by the Wigner function.

[7] To be consistent with the definition of the two-copy observable, we nevertheless introduce a factor of $\frac{1}{2}$. This ensures that $\hat{M}_z = \hat{L}_z$.

$\langle\!\langle\!\langle \hat{M} \rangle\!\rangle\!\rangle_\Psi$ stands for the expectation value on three copies of state $|\psi\rangle$. Indeed,

$$\begin{aligned}\langle\!\langle\!\langle \hat{M}_x \rangle\!\rangle\!\rangle_\Psi &= \tfrac{1}{2}\langle\psi|\langle\psi|\langle\psi|\hat{M}_x|\psi\rangle|\psi\rangle|\psi\rangle \\ &= \tfrac{1}{2}(\langle x\rangle\langle p\rangle - \langle p\rangle\langle x\rangle) = 0 \end{aligned} \quad (65)$$

and similarly for $\langle\!\langle\!\langle \hat{M}_y \rangle\!\rangle\!\rangle_\Psi$ and $\langle\!\langle\!\langle \hat{M}_z \rangle\!\rangle\!\rangle_\Psi$. The variance of $\hat{M}$ is thus equal to its second-order moment, which is computed in Appendix D. We obtain

$$\begin{aligned}(\Delta\hat{M})^2 &= \langle\!\langle\!\langle \hat{M}^2 \rangle\!\rangle\!\rangle \\ &= \tfrac{1}{3}\langle\!\langle\!\langle (M_y + M_x + M_z)^2 \rangle\!\rangle\!\rangle \\ &= \tfrac{1}{2}\langle x^2\rangle\langle p^2\rangle - \tfrac{1}{2}\langle x^2\rangle\langle p\rangle\langle p\rangle - \tfrac{1}{2}\langle p^2\rangle\langle x\rangle\langle x\rangle \\ &\quad + \tfrac{1}{2}\langle\{x,p\}\rangle\langle x\rangle\langle p\rangle - \tfrac{1}{8}\langle\{x,p\}\rangle^2 + \tfrac{1}{8}\langle[x,p]\rangle^2 \\ &= \tfrac{1}{2}\big(\det\gamma + \tfrac{1}{4}\langle[x,p]\rangle^2\big), \end{aligned} \quad (66)$$

so the variance of $\hat{M}$ is related to the determinant of the covariance matrix, in analogy with Eq. (19). Once again, since a variance is non-negative, we deduce that

$$\det\gamma \geqslant -\tfrac{1}{4}\langle[x,p]\rangle^2. \quad (67)$$

If x and p are classical, they commute and Eq. (67) expresses that a covariance matrix is always positive semidefinite. In contrast, if x and p are canonically conjugate quantum variables, they do not commute ($[x,p]=i$) and Eq. (67) implies $\det\gamma \geqslant \frac{1}{4}$, which is the Schrödinger-Robertson relation. This suggests that the three-copy operator $\hat{M}$ is a good uncertainty observable, which is invariant under all Gaussian unitaries (including displacements this time). It gives zero with certainty for all Gaussian pure states (regardless of the mean values of x and p). We define an entropic uncertainty relation based on the Shannon entropy of this observable

$$H(\hat{M})_\rho \geqslant 0. \quad (68)$$

As before, to compute the Shannon entropy of $\hat{M}$, we need to know its eigenvectors and evaluate the associated measurement probabilities. Since $\hat{M} = (\hat{M}_x + \hat{M}_y + \hat{M}_z)/\sqrt{3}$ is the component of an angular momentum in the direction $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, its eigenspectrum is well known. More precisely, the eigenvalues of $\hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2$ and $\hat{M}$ are given, respectively, by

$$\begin{aligned} l^* &= 0, \quad m = 0, \\ l^* &= \tfrac{1}{2}, \quad m = \{-\tfrac{1}{2}, 0, \tfrac{1}{2}\}, \\ l^* &= 1, \quad m = \{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\}, \end{aligned} \quad (69)$$

etc. We do not denote the squared angular momentum operator $\hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2$ simply as $\hat{M}^2$ here in order to avoid confusion with the square of our uncertainty observable $\hat{M}$ (which is a component on the angular momentum in a specific direction). Comparing to a genuine angular momentum, the eigenvalues are all divided by 2 because of the definition of the $\hat{M}_i$ [see Eq. (63)]. Moreover, the step between two subsequent eigenvalues is $\frac{1}{2}$ instead of 1 because the commutation relations are $[\hat{M}_i, \hat{M}_j] = \frac{i}{2}\epsilon_{ijk}\hat{M}_k$ (while it is $[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$ for a genuine angular momentum). The eigenfunctions of $\hat{M}$ are simply the spherical harmonics in the quadrature variables (x_1, x_2, x_3), but this form is not very convenient since they must be written in a rotated basis. Computing the probabilities of measuring the eigenvalues of $\hat{M}$ through the spherical harmonics does not seem to be an easy task, so we find it more suitable to use the physical realization of $\hat{M}$ (see Sec. III C).

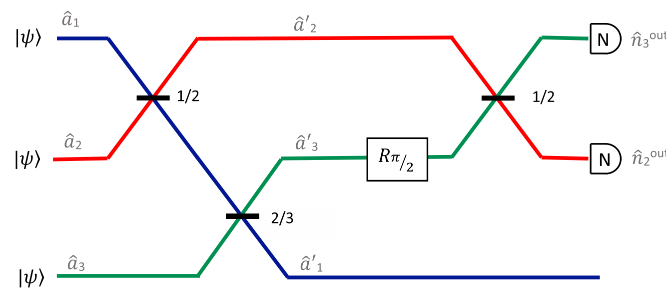

FIG. 4. Physical realization of a measurement of the three-copy uncertainty observable $\hat{M}$. Starting from three identical copies of state $|\psi\rangle$, we first apply two beam splitters (on modes 1 and 2 with transmittance $\frac{1}{2}$ and then on modes 1 and 3 with transmittance $\frac{2}{3}$). This effects a rotation in phase space such that $\hat{M}$ is rotated towards $\hat{M}_x$, which is measured by the second part of the circuit consisting of a $\pi/2$ rotation and a 50:50 beam splitter. By measuring the photon-number difference of modes 2 and 3, we thus access $\hat{M}$. The outcome is zero if and only if $|\psi\rangle$ is a minimum-uncertainty state (Gaussian pure state regardless of its position in phase space).

B. Alternative definitions

Using the relations between the x, p quadratures and the mode operators, we can express the three angular momentum components as

$$\begin{aligned} \hat{M}_x &= \frac{i}{2}(\hat{a}_2\hat{a}_3^\dagger - \hat{a}_2^\dagger\hat{a}_3), \\ \hat{M}_y &= \frac{i}{2}(\hat{a}_3\hat{a}_1^\dagger - \hat{a}_3^\dagger\hat{a}_1), \\ \hat{M}_z &= \frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2). \end{aligned} \quad (70)$$

This also allows us to express the squared angular momentum operator as

$$\begin{aligned}\hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2 = \tfrac{1}{4}\big[&(\hat{n}_1+\hat{n}_2+\hat{n}_3)(\hat{n}_1+\hat{n}_2+\hat{n}_3+1) \\ &- \big(\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2} + \hat{a}_3^{\dagger 2}\big)\big(\hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2\big)\big], \end{aligned} \quad (71)$$

where $\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$. It is symmetric in the modes, but does not have the usual $l(l+1)$ form as we had found for $\hat{L}^2$ in Eq. (30). Note also that the three components $\hat{M}_x$, $\hat{M}_y$, and $\hat{M}_z$ can be written in terms of Gell-Mann matrices, which generalize the Pauli matrices in 3×3 dimensions. This makes the counterpart to Eqs. (25) and (26) (see Appendix E).

C. Physical realization of $\hat{M}$

We show in Fig. 4 an optical circuit that allows us to measure the three-copy uncertainty observable $\hat{M}$. It is similar to the circuit for the two-copy observable $\hat{L}_z$ in the sense that, in the last stage of the circuit, we apply a $\pi/2$ rotation followed by a 50:50 beam splitter and then compute the difference between the output photon numbers. If the circuit was limited to this last stage, the photon-number difference on modes 2 and 3 would yield $\hat{M}_x$, in accordance with the first of Eqs. (70), which is analogous to Eq. (21). However, this transformation is preceded by two beam splitters of transmittance $\frac{1}{2}$ (on modes 1 and 2) and $\frac{2}{3}$ (on modes 1 and 3). The effect of these beam splitters is to make the appropriate rotation in phase space so that the direction $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is turned to (1,0,0), that is, the x direction. Indeed, after applying the two beam splitters, the mode operators are given by

$$\hat{a}'_1 = \frac{1}{\sqrt{3}}(\hat{a}_1 + \hat{a}_2 + \hat{a}_3),$$
$$\hat{a}'_2 = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2), \tag{72}$$
$$\hat{a}'_3 = \frac{1}{\sqrt{6}}(\hat{a}_1 + \hat{a}_2 - 2\hat{a}_3).$$

In particular, the first mode operator becomes the sum of the three input mode operators. This means that measuring the x-component angular momentum $\hat{M}_x$ after this rotation (i.e., on modes $\hat{a}'_1$, $\hat{a}'_2$, and $\hat{a}'_3$) yields the value of $(\hat{M}_x + \hat{M}_y + \hat{M}_z)/\sqrt{3}$ before the rotation, which is precisely the desired uncertainty observable $\hat{M}$. Therefore, keeping in mind the analogy with the two-copy observable $\hat{L}_z$, we can access $\hat{M}$ simply by applying a $\pi/2$ rotation followed by a 50:50 beam splitter on modes 2 and 3. The output photon-number difference yields

$$\hat{M} = \frac{1}{2}\left(\hat{n}_2^{\text{out}} - \hat{n}_3^{\text{out}}\right) = \frac{i}{2}(\hat{a}'_2\hat{a}'^{\dagger}_3 - \hat{a}'^{\dagger}_2\hat{a}'_3). \tag{73}$$

Interestingly, the invariance of $\hat{M}$ under displacements is easy to understand from the circuit of Fig. 4. Let us insert a displacement $D(\alpha)$ on each input state of the circuit. After the first two beam splitters, the displacement on the three modes becomes

$$D(\alpha)^{\otimes 3} \to D(\sqrt{3}\alpha)D(0)D(0). \tag{74}$$

Hence, regardless of the value of α, the displacement is zero on modes 2 and 3 just at the point where we apply the $\pi/2$ rotation and the last beam splitter. Therefore, the result of the measurement of the photon-number difference between modes 2 and 3 at the end of the circuit, which gives $\hat{M}$, does not depend on the displacement.

Note that we still have a degree of freedom in the state obtained after applying the two first beam splitters in Fig. 4. Indeed, we can easily verify that applying any real rotation in phase space between modes 2 and 3, i.e., inserting a beam splitter coupling these modes just before the second part of the circuit, does not affect M_x; hence it does not change the measured value of $\hat{M}$. This is related to the fact that $\hat{M}_x$ is invariant under a real rotation between systems 2 and 3. Indeed, if we define

$$\hat{x}'_2 = \cos\theta\hat{x}_2 + \sin\theta\hat{x}_3, \quad \hat{x}'_3 = -\sin\theta\hat{x}_2 + \cos\theta\hat{x}_3, \tag{75}$$

and similarly for the p quadratures, we can easily show that

$$\hat{M}'_x = \tfrac{1}{2}(\hat{x}'_2\hat{p}'_3 - \hat{p}'_2\hat{x}'_3) = \tfrac{1}{2}(\hat{x}_2\hat{p}_3 - \hat{p}_2\hat{x}_3) = \hat{M}_x. \tag{76}$$

D. Entropic uncertainty relation based on $\hat{M}$

It is easy to verify that our three-copy uncertainty observable vanishes on any pure Gaussian state, i.e., squeezed coherent state. If we insert three copies of a squeezed coherent state in the optical circuit of Fig. 4, we obtain after the first two beam splitters the same three squeezed coherent states (albeit with changed mean values, as explained earlier).[8] This means that, similarly to the two-copy case, we get a zero photon-number difference with probability one at the output of the circuit. Consequently, the entropy of $\hat{M}$ is equal to zero for any pure Gaussian state. Our entropic uncertainty relation $H(\hat{M}) \geqslant 0$ thus admits the exact same set of minimum-uncertainty states as the Schrödinger-Robertson uncertainty relation.

Furthermore, it appears that the entropic uncertainty relation based on $\hat{M}$ coincides with the one based on $\hat{L}_z$ in the special case of Gaussian states centered at the origin. Indeed, if we plug in three copies of an arbitrary Gaussian state, pure or mixed, at the input of the circuit of Fig. 4, we again get the same three Gaussian states after the first two beam splitters (albeit with changed mean values). In particular, we find two copies of the input Gaussian state on modes 2 and 3 (albeit centered on the origin). Since the rest of the circuit is the same as the two-copy circuit of Fig. 1, all conclusions we had drawn for $\hat{L}_z$ hold for $\hat{M}$ too. In particular, the entropy of a Gaussian state will be the same, namely,

$$H(\hat{M})_{\rho^G} = H(\hat{L}_z)_{\rho^G}, \tag{77}$$

with $H(\hat{L}_z)_{\rho^G}$ defined in Eq. (59).

In the case of non-Gaussian states centered at the origin, however, we expect the entropy $H(\hat{M})$ to deviate from $H(\hat{L}_z)$, so it seems relevant to define a distinct entropic uncertainty relation $H(\hat{M}) \geqslant 0$. For example, if we insert three copies of Fock state $|1\rangle$ in the circuit of Fig. 4, the state of modes 2 and 3 differs from $|1\rangle^{\otimes 2}$ after the first two beam splitters, so the second part of the circuit acts differently. Hence, the entropy of the three-copy observable $H(\hat{M})_{|1\rangle}$ differs from that of the two-copy observable $H(\hat{L}_z)_{|1\rangle}$ (as computed in Appendix C).

IV. CONCLUSION

We have paved the way towards the construction of entropic uncertainty relations for continuous-variable bosonic states that are invariant under Gaussian unitary transformations (rotation, squeezing, and displacement in phase space). This was achieved by defining the notion of a multicopy uncertainty observable (especially a two-copy observable $\hat{L}_z$ and a three-copy observable $\hat{M}$) with ingrained invariance, building on the Schwinger representation of angular momenta in terms of bosonic operators. Observable $\hat{L}_z$ acts on two replicas of a continuous-variable state and is invariant under rotation and squeezing (so it is relevant for states centered on the origin only), while $\hat{M}$ acts on three replicas and exhibits extra invariance under displacement (so it is relevant for

[8]If the product of two identical Gaussian states impinges on a beam splitter, we obtain at the output a product of two Gaussian states with the same covariance matrix (only the mean values are changed).

any state). Expressing the non-negativity of the variance of both (discrete-spectrum) observables $\hat{L}_z$ and $\hat{M}$ leads to the Schrödinger-Robertson uncertainty relation, which supports the fact that these observables capture uncertainty in phase space (or the deviation from pure Gaussianity). Based on this, we have constructed two entropic uncertainty relations by expressing the fact that the Shannon entropy of $\hat{L}_z$ and $\hat{M}$ must be non-negative for any physical state. Given the intrinsic invariance of $\hat{L}_z$ and $\hat{M}$, these entropic uncertainty relations are automatically invariant under Gaussian unitaries and are saturated by all pure Gaussian states (with $\hat{L}_z$ being restricted to states centered on the origin). In some sense, they can be viewed as the entropic counterpart to the Schrödinger-Robertson uncertainty relation.

Although such a Gaussian invariance is not strictly necessary for a measure of uncertainty to be meaningful, if the purpose is to define a measure of uncertainty in phase space rather than a function merely relating the uncertainties of variables x and p, it is natural to require this measure to be invariant under symplectic transformations, which leave the volume in phase space invariant. Remarkably, it is the angular momentum algebra of the uncertainty observables $\hat{L}_z$ and $\hat{M}$ that ensures this invariance in our construction.

We have described optical circuits enabling us to measure the observable $\hat{L}_z$ ($\hat{M}$) starting from two (three) replicas of the input state. From an experimental perspective, measuring these observables requires the preparation of two (three) identical replicas of an optical state, followed by a linear-optics circuit combining them in order to achieve a specific joint measurement. Thus, the identical optical states should be generated from the same laser (to share the same phase reference) and interferometric stability should be ensured in the optical circuit up to the final measurement of the photon-number difference. The complexity of such a setup is comparable to that of various current experiments on multiphoton interference effects in multimode circuits (in particular those based on integrated photonic chips; see, e.g., [16]), so it seems reasonable to access the uncertainty $\hat{L}_z$ or $\hat{M}$ of a state in x-p space, at least when dealing with the optical analogs of x and p.

Regardless of the experimental feasibility of measuring observable $\hat{L}_z$ or $\hat{M}$, the sole theoretical definition of these optical circuits proved to be useful in deriving a closed formula for the Shannon entropy $H(\hat{L}_z)$ or $H(\hat{M})$ in the special case of Gaussian states (both entropies coincide in that case). However, we have not found a simple method to compute these entropies for non-Gaussian states, which we leave as a topic for further study. Another problem that we leave open is to find an operational meaning for $H(\hat{L}_z)$ and $H(\hat{M})$, which would help in interpreting physically the associated entropic uncertainty relations. It is fascinating that the Shannon entropy of a (discrete-spectrum) angular momentum observable such as $\hat{L}_z$ or $\hat{M}$ can be connected to the differential entropy of the Wigner function in (continuous-variable) x-p space, at least for Gaussian states.

Furthermore, an interesting issue raised by this work is to elucidate the reason why three replicas seem to be necessary to build an uncertainty observable that possesses the desired invariance. Since the left-hand side of the Schrödinger-Robertson relation is quartic in the position-momentum variables, the variance of a two-copy observable might have been sufficient (assuming the observable is linear in the quadrature variables of each copy) and it is unclear why we had to consider the variance of a three-copy observable instead (this could in principle give access to sixth-order moments in x and p). Conversely, a four-copy observable may also have been considered, where some constraint on its mean (e.g., the observable must be positive semidefinite) instead of its variance would induce an uncertainty relation. More generally, a valuable extension of this work would be to investigate general multicopy uncertainty observables.

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APPENDIX A: CALCULATION OF THE COMMUTATOR BETWEEN $\hat{L}_x$ AND $\hat{L}_y$

Let us show that the two-copy operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ obey the commutation relations for angular momenta. As an example, we calculate the commutator between $\hat{L}_x$ and $\hat{L}_y$ using the properties of Pauli matrices, namely,

$$[\hat{L}_x, \hat{L}_y] = \tfrac{1}{4}[A^\dagger \sigma_z A, A^\dagger \sigma_x A] = \tfrac{1}{4}A^\dagger(\sigma_z A A^\dagger \sigma_x - \sigma_x A A^\dagger \sigma_z)A. \tag{A1}$$

where $\hat{A} = \binom{\hat{a}_1}{\hat{a}_2}$. We can easily compute

$$AA^\dagger = (\hat{L}_0 + 1)\mathbb{1} + \hat{L}_y\sigma_x + \hat{L}_z\sigma_y + \hat{L}_x\sigma_z, \tag{A2}$$

where

$$\hat{L}_0 = \frac{\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2}{2} = \frac{1}{2}A^\dagger A, \tag{A3}$$

so the commutator becomes

$$\begin{aligned}[\hat{L}_x, \hat{L}_y] &= \frac{1}{4}A^\dagger\{\sigma_z[(\hat{L}_0 + 1)\mathbb{1} + \hat{L}_y\sigma_x + \hat{L}_z\sigma_y + \hat{L}_x\sigma_z]\sigma_x \\ &\quad - \sigma_x[(\hat{L}_0 + 1)\mathbb{1} + \hat{L}_y\sigma_x + \hat{L}_z\sigma_y + \hat{L}_x\sigma_z]\sigma_z\}A \\ &= \frac{1}{4}A^\dagger\{(\hat{L}_0 + 1)[\sigma_z, \sigma_x] - 2i\hat{L}_z\}A \\ &= \frac{i}{2}A^\dagger[(\hat{L}_0 + 1)\sigma_y - \hat{L}_z]A \\ &= \frac{i}{2}A^\dagger\big[\big(\tfrac{1}{2}A^\dagger A + 1\big)\sigma_y - \tfrac{1}{2}A^\dagger\sigma_y A\big]A \\ &= \frac{i}{2}A^\dagger\sigma_y A + \frac{i}{4}[A^\dagger(A^\dagger A)\sigma_y A - A^\dagger(A^\dagger\sigma_y A)A] \\ &= i\hat{L}_z + \frac{i}{4}[A^\dagger(A^\dagger A)\sigma_y A - A^\dagger(A^\dagger\sigma_y A)A]. \end{aligned} \tag{A4}$$

Now we just need to show that the last term in this expression is equal to zero. However, the calculation is not straightforward because the matrices do not all have consistent

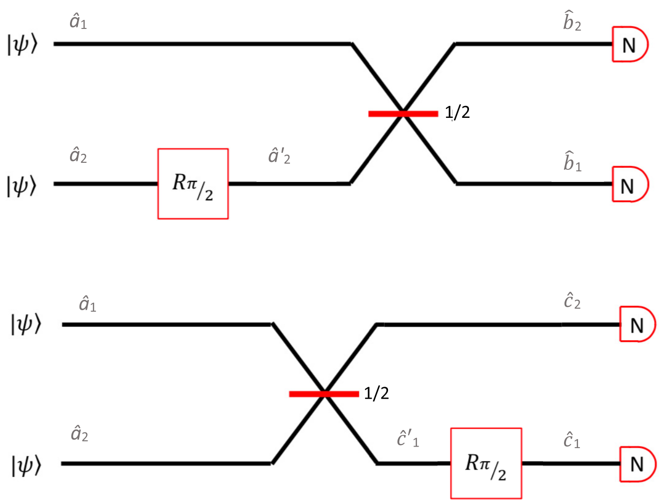

FIG. 5. Physical realization of a measurement of $\hat{L}_z$ (upper circuit) and $\hat{L}_y$ (lower circuit) starting from two identical copies of state $|\psi\rangle$. The input mode operators ($\hat{a}_1$ and $\hat{a}_2$) are transformed into the output mode operators ($\hat{b}_1$ and $\hat{b}_2$) in the upper circuit, consisting in a $\pi/2$ phase rotation followed by a 50:50 beam splitter. The photon-number difference yields $\hat{L}_z$. Interchanging the $\pi/2$ phase rotation and 50:50 beam splitter leads to the output mode operators ($\hat{c}_1$ and $\hat{c}_2$) in the lower circuit, so the photon-number difference yields $\hat{L}_y$. The input photon-number difference simply yields $\hat{L}_x$.

dimensions for multiplication.[9] Nevertheless, we can prove that

$$A^\dagger M(A^\dagger A)NA = \sum_{ijk} \hat{a}_i^\dagger M_{ij}\left(\sum_l \hat{a}_l^\dagger \hat{a}_l\right)N_{jk}\hat{a}_k$$
$$= \sum_l \hat{a}_l^\dagger\left(\sum_{ijk}\hat{a}_i^\dagger M_{ij}N_{jk}\hat{a}_k\right)\hat{a}_l$$
$$= A^\dagger(A^\dagger MNA)A, \qquad \text{(A5)}$$

where the objects inside the parentheses have the dimension of a scalar and the matrices M and N are composed of scalar numbers, so they commute with the mode operators. If we define $M = \mathbb{1}$ and $N = \sigma_y$, we have

$$A^\dagger(A^\dagger A)\sigma_y A - A^\dagger(A^\dagger\sigma_y A)A = 0, \qquad \text{(A6)}$$

which completes the calculation of the commutator

$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z. \qquad \text{(A7)}$$

The other commutators can be calculated similarly.

APPENDIX B: ALTERNATIVE DEFINITIONS OF $(\hat{L}_x, \hat{L}_y, \hat{L}_z)$

The angular momentum components $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ can be expressed in several ways as a function of the input mode operators $(\hat{a}_1, \hat{a}_2)$ or output mode operators $(\hat{b}_1, \hat{b}_2)$ of the circuit depicted in Fig. 1, or even the output mode operators $(\hat{c}_1, \hat{c}_2)$ of another circuit. This is explained in Fig. 5, where the first circuit is the same as in Fig. 1. In the second circuit shown in Fig. 5, the $\pi/2$ phase rotation is applied after the 50:50 beam splitter transformation, and the output mode operators are labeled as $\hat{c}_1$ and $\hat{c}_2$. The mode operators evolve as

$$\hat{a}_1 \to \hat{c}_1' = (\hat{a}_1 + \hat{a}_2)/\sqrt{2}, \quad \hat{a}_2 \to \hat{c}_2 = (\hat{a}_1 - \hat{a}_2)/\sqrt{2},$$
$$\hat{c}_1' \to \hat{c}_1 = -i\hat{c}_1'. \qquad \text{(B1)}$$

Let us show that the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ can equivalently be expressed in terms of the $\hat{a}$, $\hat{b}$, or $\hat{c}$ mode operators. In terms of the mode operator $\hat{a}$, the expressions are given by Eqs. (21) and (27). Using the first circuit, we already showed that $\hat{L}_z$ corresponds to one-half the photon-number difference of the output modes [see Eq. (24)], and it is easy to show that

$$\hat{L}_x = \frac{1}{2}(\hat{b}_1\hat{b}_2^\dagger + \hat{b}_1^\dagger\hat{b}_2), \quad \hat{L}_y = \frac{i}{2}(\hat{b}_1\hat{b}_2^\dagger - \hat{b}_1^\dagger\hat{b}_2). \qquad \text{(B2)}$$

Based on the second circuit, we can do similar calculations to express $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in terms of the mode operators $\hat{c}$. The results are summarized in Table I, which also exhibits the expressions of $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ in terms of the quadrature operators (first row). Moreover, we have

$$\hat{L}_z = \tfrac{1}{2}\hat{A}^\dagger\sigma_y\hat{A} = \tfrac{1}{2}\hat{B}^\dagger\sigma_z\hat{B} = \tfrac{1}{2}\hat{C}^\dagger\sigma_x\hat{C}$$
$$\hat{L}_y = \tfrac{1}{2}\hat{A}^\dagger\sigma_x\hat{A} = \tfrac{1}{2}\hat{B}^\dagger\sigma_y\hat{B} = \tfrac{1}{2}\hat{C}^\dagger\sigma_z\hat{C}$$
$$\hat{L}_x = \tfrac{1}{2}\hat{A}^\dagger\sigma_z\hat{A} = \tfrac{1}{2}\hat{B}^\dagger\sigma_x\hat{B} = \tfrac{1}{2}\hat{C}^\dagger\sigma_y\hat{C} \qquad \text{(B3)}$$

where $\hat{A} = \binom{\hat{a}_1}{\hat{a}_2}$, $\hat{B} = \binom{\hat{b}_1}{\hat{b}_2}$ and $\hat{C} = \binom{\hat{c}_1}{\hat{c}_2}$.

TABLE I. All possible definitions of the operators $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ in terms of the mode operators $(\hat{a}_1, \hat{a}_2)$, $(\hat{b}_1, \hat{b}_2)$, and $(\hat{c}_1, \hat{c}_2)$ and quadrature operators $(\hat{x}, \hat{p})$.

	$\hat{L}_x$	$\hat{L}_y$	$\hat{L}_z$
$\hat{x}, \hat{p}$	$\frac{(\hat{x}_1^2+\hat{p}_1^2)-(\hat{x}_2^2+\hat{p}_2^2)}{4}$	$\frac{1}{2}(\hat{x}_1\hat{x}_2 + \hat{p}_1\hat{p}_2)$	$\frac{1}{2}(\hat{x}_1\hat{p}_2 - \hat{p}_1\hat{x}_2)$
$\hat{a}, \hat{a}^\dagger$	$\frac{1}{2}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)$	$\frac{1}{2}(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger)$	$\frac{i}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2)$
$\hat{b}, \hat{b}^\dagger$	$\frac{1}{2}(\hat{b}_1\hat{b}_2^\dagger + \hat{b}_1^\dagger\hat{b}_2)$	$\frac{i}{2}(\hat{b}_1\hat{b}_2^\dagger - \hat{b}_1^\dagger\hat{b}_2)$	$\frac{1}{2}(\hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2)$
$\hat{c}, \hat{c}^\dagger$	$\frac{i}{2}(\hat{c}_1\hat{c}_2^\dagger - \hat{c}_1^\dagger\hat{c}_2)$	$\frac{1}{2}(\hat{c}_1^\dagger\hat{c}_1 - \hat{c}_2^\dagger\hat{c}_2)$	$\frac{1}{2}(\hat{c}_1\hat{c}_2^\dagger + \hat{c}_1^\dagger\hat{c}_2)$

APPENDIX C: CALCULATION OF $H(\hat{L}_z)$ FOR SOME EXAMPLES OF NON-GAUSSIAN STATES

We compute here the entropy of our two-copy uncertainty observable $\hat{L}_z$ for some examples of non-Gaussian states.

Example 1. Consider the Fock state $|1\rangle$. If we insert two copies of $|1\rangle$ in the optical circuit of Fig. 1, we find the state

$$\frac{1}{\sqrt{2}}(|0\,2\rangle - |2\,0\rangle) \qquad \text{(C1)}$$

at the output. Therefore, the photon-number difference will be ± 2, each with probability $\frac{1}{2}$, and the entropy of $\hat{L}_z$ is given by

$$H(\hat{L}_z)_{|1\rangle} = -\sum_m p_m \ln p_m = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2. \qquad \text{(C2)}$$

[9]The matrix multiplication is associative only if we multiply matrices of dimensions $n \times m$, $m \times p$, and $p \times q$.

As expected, this value is greater than zero since we are dealing with a non-Gaussian state, in agreement with our entropic uncertainty relation (47).

Example 2. Consider now a mixture of $|0\rangle$ and $|1\rangle$,

$$\rho = \alpha|0\rangle\langle 0| + (1-\alpha)|1\rangle\langle 1|. \tag{C3}$$

Here we do not use the optical circuit to compute the entropy, but rather Eq. (46), namely,[10]

$$p_m = \sum_{l=|m|}^{\infty} \langle\!\langle l, m\|\rho\otimes\rho\|l, m\rangle\!\rangle. \tag{C4}$$

Since

$$\rho\otimes\rho = \alpha^2|00\rangle\langle 00| + (1-\alpha)^2|11\rangle\langle 11| + \alpha(1-\alpha)|01\rangle\langle 01| + \alpha(1-\alpha)|10\rangle\langle 10|, \tag{C5}$$

we only need to consider states $\|l, m\rangle\!\rangle$ with $l = \{0, \frac{1}{2}, 1\}$, which are given in Eqs. (34)-(36). Accordingly, the possible values of m are $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. We can now compute the different probabilities p_m,

$$p_0 = \sum_{l=0}^{1} \langle\!\langle l, 0\|\rho\otimes\rho\|l, 0\rangle\!\rangle = \langle\!\langle 0, 0\|\rho\otimes\rho\|0, 0\rangle\!\rangle + \langle\!\langle 1, 0\|\rho\otimes\rho\|1, 0\rangle\!\rangle = \alpha^2,$$

$$p_{\pm 1/2} = \sum_{l=0}^{1} \langle\!\langle l, \pm\tfrac{1}{2}\|\rho\otimes\rho\|l, \pm\tfrac{1}{2}\rangle\!\rangle = \langle\!\langle \tfrac{1}{2}, \pm\tfrac{1}{2}\|\rho\otimes\rho\|\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle\!\rangle = \alpha(1-\alpha),$$

$$p_{\pm 1} = \sum_{l=0}^{1} \langle\!\langle l, \pm 1\|\rho\otimes\rho\|l, \pm 1\rangle\!\rangle = \langle\!\langle 1, \pm 1\|\rho\otimes\rho\|1, \pm 1\rangle\!\rangle = \frac{(1-\alpha)^2}{2} \tag{C6}$$

and the entropy of $\hat{L}_z$ is equal to

$$H(\hat{L}_z)_\rho = (1-\alpha)^2\ln 2 - 2\alpha\ln\alpha - 2(1-\alpha)\ln(1-\alpha), \tag{C7}$$

which is always greater than zero except when $\alpha = 1$ because then ρ is simply equal to the vacuum state. If $\alpha = 0$, we find $H(\hat{L}_z)_\rho = \ln 2$ as expected from Example 1.

Note that the Shannon entropy of this mixture is a concave function of α. This suggests that $H(\hat{L}_z)$ is probably a concave function in general.

APPENDIX D: DERIVATION OF THE SECOND-ORDER MOMENT OF $\hat{M}$

To compute the second-order moment of the three-copy observable $\hat{M}$, we first note that

$$(M_x + M_y + M_z)^2 = M_x^2 + M_y^2 + M_z^2 + M_xM_y + M_yM_x + M_xM_z + M_zM_x + M_yM_z + M_zM_y, \tag{D1}$$

with

$$\begin{aligned} M_x^2 + M_y^2 + M_z^2 &= \tfrac{1}{4}\big(x_2^2p_1^2 + x_3^2p_1^2 + x_1^2p_2^2 + x_3^2p_2^2 + x_1^2p_3^2 + x_2^2p_3^2\big) - \tfrac{1}{4}(x_1p_1p_2x_2 + x_2p_2p_3x_3 + x_3p_3p_1x_1 + p_1x_1x_2p_2 + p_2x_2x_3p_3 + p_3x_3x_1p_1) \\ &= \tfrac{1}{4}\big(x_2^2p_1^2 + x_3^2p_1^2 + x_1^2p_2^2 + x_3^2p_2^2 + x_1^2p_3^2 + x_2^2p_3^2\big) - \tfrac{1}{8}(\{x_1, p_1\}\{x_2, p_2\} + \{x_2, p_2\}\{x_3, p_3\} + \{x_3, p_3\}\{x_1, p_1\}) \\ &\quad + \tfrac{1}{8}([x_1, p_1][x_2, p_2] + [x_2, p_2][x_3, p_3] + [x_3, p_3][x_1, p_1]) \end{aligned} \tag{D2}$$

and

$$\begin{aligned} M_xM_y + M_xM_z + M_yM_z + M_yM_x + M_zM_x + M_zM_y = &-\tfrac{1}{2}\big(p_1p_2x_3^2 + p_1p_3x_2^2 + p_2p_3x_1^2\big) \\ &- \tfrac{1}{2}\big(p_1^2x_2x_3 + p_2^2x_1x_3 + p_3^2x_1x_2\big) + \tfrac{1}{4}\big(\{x_1, p_1\}(x_2p_3 + p_2x_3) \\ &+ \{x_2, p_2\}(x_1p_3 + p_1x_3) + \{x_3, p_3\}(x_1p_2 + p_1x_2)\big). \end{aligned} \tag{D3}$$

Therefore, if we take the mean value of $\hat{M}^2$ on three copies of the state we obtain

$$\begin{aligned} \langle\!\langle\!\langle \hat{M}^2\rangle\!\rangle\!\rangle &= \tfrac{1}{3}\langle\!\langle\!\langle (M_y + M_x + M_z)^2\rangle\!\rangle\!\rangle \\ &= \tfrac{1}{12}6\langle x^2\rangle\langle p^2\rangle - \tfrac{1}{6}3\langle x^2\rangle\langle p\rangle\langle p\rangle - \tfrac{1}{6}3\langle p^2\rangle\langle x\rangle\langle x\rangle + \tfrac{1}{12}6\langle\{x, p\}\rangle\langle x\rangle\langle p\rangle - \tfrac{1}{24}3\langle\{x, p\}\rangle^2 + \tfrac{1}{24}3\langle[x, p]\rangle^2 \\ &= \tfrac{1}{2}\big(\det\gamma + \tfrac{1}{4}\langle[x, p]\rangle^2\big). \end{aligned} \tag{D4}$$

APPENDIX E: EXPRESSION OF $\hat{M}_x$, $\hat{M}_y$, AND $\hat{M}_z$ IN TERMS OF GELL-MANN MATRICES

Another way of defining the three angular momentum components $\hat{M}_x$, $\hat{M}_y$, and $\hat{M}_z$ relies on the Gell-Mann matrices, which generalize the Pauli matrices in 3×3 dimensions. There are eight Gell-Mann matrices, denoted by λ_i, but we only need three of them, namely,

$$S_x \equiv \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y \equiv -\lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z \equiv \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{E1}$$

[10] Note that there is a slight abuse of notation here since the sum on l takes half-integer steps, that is, $l = \{|m|, |m| + \frac{1}{2}, |m| + 1, \ldots\}$.

In analogy with Eqs. (25) and (26), we can write the three operators $\hat{M}_i$ as

$$\hat{M}_x = \tfrac{1}{2}A^\dagger S_x A, \quad \hat{M}_y = \tfrac{1}{2}A^\dagger S_y A, \quad \hat{M}_z = \tfrac{1}{2}A^\dagger S_z A, \tag{E2}$$

where we have defined $\hat{A} = (\hat{a}_1 \quad \hat{a}_2 \quad \hat{a}_3)^T$. From this formulation, we can easily compute the commutation relations between the $\hat{M}_i$ observables. They almost obey those of an angular momentum, that is,

$$[\hat{M}_i, \hat{M}_j] = \frac{i}{2}\epsilon_{ijk}\hat{M}_k, \tag{E3}$$

where the $\frac{1}{2}$ factor comes from our definition of the $\hat{M}_i$ as already mentioned. All the algebraic properties of operators $\hat{M}_x$, $\hat{M}_y$, and $\hat{M}_z$ should be describable in a unified form based on (E2) and the properties of the Gell-Mann matrices.

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